

On the Schrödinger group

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Abstract

In this paper, we compute the Leibniz homology of the Schrödinger algebra. We show that it is a graded vector space generated by tensors in dimensions $2n - 2$ and $2n$. The Leibniz homology of the full Galilei algebra is also calculated.

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1 Introduction

Leibniz homology was introduced by Jean-Louis Loday (see [7, 10.6]) as a non commutative version of Lie algebra homology. In this paper, we calculate this homology for one of the most important non semisimple Lie algebra of mathematical physics. Thanks to its semidirect sum structure, the Schrödinger algebra is presented as an abelian extension of a semisimple Lie algebra. This enables us to calculate its Leibniz (co)homology using techniques previously applied on the Lie algebra of the euclidean group [1], the affine symplectic Lie algebra [9] and the Poincaré algebra [2]. Recall that the (non centrally extended or massless) full Galilei group $\widetilde{GAL}(n)$ of n -dimensional space consists of real $(n + 2) \times (n + 2)$ matrices

$$\begin{bmatrix} \mathcal{X} & v & a \\ 0 & A_n & B_n \\ 0 & C_n & D_n \end{bmatrix} \quad (M.1)$$

with $\mathcal{X} \in O(n; \mathbb{R})$, $a, v \in \mathbb{R}^n$ and $A_n, B_n, C_n, D_n \in \mathbb{R}$ with $A_n D_n - B_n C_n \neq 0$. Its group structure is the semidirect product

$$\widetilde{GAL}(n) = (O(n; \mathbb{R}) \times GL(2; \mathbb{R})) \ltimes (\mathbb{R}^n \times \mathbb{R}^n).$$

With the condition $A_n D_n - B_n C_n = 1$, the matrices (M.1) above constitute the Schrödinger group $Sch(n)$. Its Lie algebra \mathfrak{sch}_n is an abelian extension of the Lie algebra

$$\bar{\mathfrak{h}}_n = \mathfrak{so}(n; \mathbb{R}) \oplus \mathfrak{sl}(2; \mathbb{R}).$$

We calculate its Leibniz homology and obtain the isomorphism of graded vector spaces

$$HL_*(\mathfrak{sch}_n; \mathbb{R}) \cong (\mathbb{R} \oplus \langle \tilde{\zeta}_n \rangle \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n),$$

where $\langle \tilde{\zeta}_n \rangle$ denotes a 1-dimensional vector space in dimension $2n - 2$ generated by the \mathfrak{sch}_n -invariant

$$\tilde{\zeta}_n = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n-2}} \text{sgn}(\sigma) y_{\sigma(1)} \otimes \dots \widehat{y_{\sigma(i)}} \dots \otimes y_{\sigma(n)} \otimes y_{\sigma(n+1)} \otimes \dots \widehat{y_{\sigma(n+i)}} \dots \otimes y_{\sigma(2n)} \quad ,$$

$\langle \tilde{\alpha}_n \rangle$ denotes a 1-dimensional vector space in dimension $2n$ generated by the \mathfrak{sch}_n -invariant

$$\tilde{\alpha}_n = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) y_{\sigma(1)} \otimes \dots \otimes y_{\sigma(n)} \otimes y_{\sigma(n+1)} \otimes \dots \otimes y_{\sigma(2n)}$$

and $T^*(\tilde{\gamma}_n)$ denotes the tensor algebra on a $(2n - 2)$ -degree generator which is an antisymmetrization of

$$\begin{aligned} \gamma_n = & \sum_{1 \leq i < j \leq n} X_{ij} \otimes (y_1 \wedge \dots \widehat{y_i} \dots \wedge y_n \wedge y_{n+1} \wedge \dots \widehat{y_{n+j}} \dots \wedge y_{2n}) \\ & - \sum_{1 \leq i < j \leq n} X_{ij} \otimes (y_1 \wedge \dots \widehat{y_j} \dots \wedge y_n \wedge y_{n+1} \wedge \dots \widehat{y_{n+i}} \dots \wedge y_{2n}) \end{aligned}$$

with $y_i := x_i \frac{\partial}{\partial x^{n+1}}$ and $y_{n+i} := x_i \frac{\partial}{\partial x^{n+2}}$ for $1 \leq i \leq n$.

Since $\mathfrak{gl}(2; \mathbb{R}) \cong \mathfrak{sl}(2; \mathbb{R}) \oplus D$ where $D = \left\{ \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}, d \in \mathbb{R} \right\}$, it follows that $\widetilde{\mathfrak{gal}}_n \cong \mathfrak{sch}_n \oplus D$.

The Leibniz homology of $\widetilde{\mathfrak{gal}}_n$ is then obtained via a Künneth-style formula for the homology of Leibniz algebras [8].

2 The Leibniz Homology

Recall that for any Lie algebra \mathfrak{g} over a ring k and V any \mathfrak{g} -module, the Lie algebra homology of \mathfrak{g} with coefficients in the module V , written $H_*^{Lie}(\mathfrak{g}; V)$, is the homology of the Chevalley-Eilenberg complex $V \otimes \wedge^*(\mathfrak{g})$, namely

$$V \xleftarrow{d} V \otimes \mathfrak{g}^{\wedge^1} \xleftarrow{d} V \otimes \mathfrak{g}^{\wedge^2} \xleftarrow{d} \dots \xleftarrow{d} V \otimes \mathfrak{g}^{\wedge^{n-1}} \xleftarrow{d} V \otimes \mathfrak{g}^{\wedge^n} \leftarrow \dots$$

where \mathfrak{g}^{\wedge^n} is the n th exterior power of \mathfrak{g} over k , and where

$$\begin{aligned} d(v \otimes g_1 \wedge \dots \wedge g_n) = & \sum_{1 \leq j \leq n} (-1)^j [v, g_j] \otimes g_1 \wedge \dots \widehat{g_j} \dots \wedge g_n \\ & + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} v \otimes [g_i, g_j] \wedge g_1 \wedge \dots \widehat{g_i} \dots \widehat{g_j} \dots \wedge g_n \quad [3] \end{aligned}$$

where $\widehat{g_i}$ means that the variable g_i is deleted.

For each n , we have the canonical projection $\pi : \mathfrak{g} \otimes \mathfrak{g}^{\wedge n} \longrightarrow \mathfrak{g}^{\wedge n+1}$. This gives a map of chain complexes $\mathfrak{g} \otimes \wedge^*(\mathfrak{g}) \longrightarrow \wedge^{*+1}(\mathfrak{g})$ and thus induces a k -linear map on homology

$$\pi_* : H_*^{Lie}(\mathfrak{g}; \mathfrak{g}) \longrightarrow H_{*+1}^{Lie}(\mathfrak{g}; k).$$

Recall that for any \mathfrak{g} -module M , the submodule $M^{\mathfrak{g}}$ of \mathfrak{g} -invariants is defined by

$$M^{\mathfrak{g}} = \{m \in M \mid [m, g] = 0 \text{ for all } g \in \mathfrak{g}\}.$$

Lemma 2.1. *Let*

$$0 \longrightarrow \mathfrak{J} \xrightarrow{i} \mathfrak{g}_e \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

be an abelian extension of a (semi)-simple Lie algebra \mathfrak{g} over \mathbb{R} . Then the following are natural vector space isomorphisms

$$H_*^{Lie}(\mathfrak{g}_e; \mathbb{R}) \cong H_*^{Lie}(\mathfrak{g}; \mathbb{R}) \otimes [\wedge^*(\mathfrak{J})]^{\mathfrak{g}},$$

$$H_*^{Lie}(\mathfrak{g}_e; \mathfrak{g}_e) \cong H_*^{Lie}(\mathfrak{g}; \mathbb{R}) \otimes [H_*^{Lie}(\mathfrak{J}; \mathfrak{g}_e)]^{\mathfrak{g}}.$$

Proof. The proof consists of applying the Hochschild-Serre spectral sequence to the (semi)-simple Lie algebra \mathfrak{g} , subalgebra of \mathfrak{g}_e . See [10, lemma 2.1] for details when \mathfrak{g} is simple. \square

Remark 2.2. *The (co)homology groups of $\mathfrak{sl}(2; \mathbb{R})$ and $\mathfrak{so}(n; \mathbb{R})$ are known (see [5, p.1742]). So by lemma 2.1, to determine $H_*^{Lie}(\mathfrak{sch}_n; \mathbb{R})$ and $H_*^{Lie}(\mathfrak{sch}_n; \mathfrak{sch}_n)$, it is enough to determine the appropriate modules of \mathfrak{g} -invariants.*

Recall also that for a Leibniz algebra (Lie algebra in particular) \mathfrak{g} , the Leibniz homology of \mathfrak{g} with coefficients in \mathbb{R} denoted $HL_*(\mathfrak{g}, \mathbb{R})$, is the homology of the Loday complex $T^*(\mathfrak{g})$, namely

$$\mathbb{R} \xleftarrow{0} \mathfrak{g} \xleftarrow{[\cdot, \cdot]} \mathfrak{g}^{\otimes 2} \xleftarrow{d} \dots \xleftarrow{d} \mathfrak{g}^{\otimes n-1} \xleftarrow{d} \mathfrak{g}^{\otimes n} \xleftarrow{\dots}$$

where $\mathfrak{g}^{\otimes n}$ is the n th tensor power of \mathfrak{g} over \mathbb{R} , and where

$$d(g_1 \otimes g_2 \otimes \dots \otimes g_n) = \sum_{1 \leq i < j \leq n} (-1)^j g_1 \otimes g_2 \otimes \dots \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes \dots \otimes \widehat{g_j} \dots \otimes g_n \quad [7].$$

Note that the latter complex above is infinite, so the calculation of these homology groups is most likely possible only through the use of a spectral sequence. Pirashvili [11] introduced a spectral sequence for this purpose and Lodder [10] used it to establish a structure theorem useful to determine the Leibniz homology groups of abelian extensions of (semi)-simple Lie algebras in terms of these Lie algebras invariants.

3 Leibniz homology of the Schrödinger algebra

Assume that \mathbb{R}^n is given the coordinates (x_1, x_2, \dots, x_n) , and let $\frac{\partial}{\partial x^i}$ be the unit vector fields parallel to the x_i axes respectively. It is easy to show that the Lie algebra generated by the family \bar{B}_1 below of vector fields (endowed with the bracket of vector fields) is isomorphic to $\bar{\mathfrak{h}}_n$:

$$\bar{B}_1 = \{X_{ij}, a_n, b_n, c_n\}$$

where

$$X_{ij} := -x_i \frac{\partial}{\partial x^j} + x_j \frac{\partial}{\partial x^i} \quad 1 \leq i < j \leq n,$$

$$a_n := -x_{n+1} \frac{\partial}{\partial x^{n+1}} + x_{n+2} \frac{\partial}{\partial x^{n+2}},$$

$$b_n := x_{n+1} \frac{\partial}{\partial x^{n+2}},$$

$$c_n := -x_{n+2} \frac{\partial}{\partial x^{n+1}}.$$

The brackets relations of the Lie algebra $\bar{\mathfrak{h}}_n$ are:

$$[X_{ij}, X_{ik}] = X_{jk}, \quad [X_{ij}, a_n] = 0, \quad [X_{ij}, b_n] = 0, \quad [X_{ij}, c_n] = 0,$$

$$[a_n, b_n] = -2b_n, \quad [a_n, c_n] = 2c_n, \quad [b_n, c_n] = a_n.$$

Remark 3.1. The brackets above yield the following Lie algebra isomorphisms

$$\mathfrak{sl}(2; \mathbb{R}) \cong \text{span}\{a_n, b_n, c_n\},$$

$$\mathfrak{so}(n; \mathbb{R}) \cong \text{span}\{X_{ij}, 1 \leq i < j \leq n\}.$$

Denote by \mathfrak{I}_n the Lie algebra of $\mathbb{R}^n \times \mathbb{R}^n$. The following is a vector space basis of \mathfrak{I}_n .

$$B_2 = \{x_i \frac{\partial}{\partial x^{n+1}}, x_i \frac{\partial}{\partial x^{n+2}} \mid 1 \leq i \leq n\}.$$

Then the Schrödinger algebra \mathfrak{sch}_n has an \mathbb{R} -vector space basis $\bar{B}_1 \cup B_2$ and there is a short exact sequence of Lie algebras [6, p.203]

$$0 \longrightarrow \mathfrak{I}_n \xrightarrow{i} \mathfrak{sch}_n \xrightarrow{\pi} \bar{\mathfrak{h}}_n \longrightarrow 0$$

where i is the inclusion map and π is the projection $\mathfrak{sch}_n \longrightarrow (\mathfrak{sch}_n / \mathfrak{I}_n) \cong \bar{\mathfrak{h}}_n$. Note that here, \mathfrak{I}_n is the standard representation of $\bar{\mathfrak{h}}_n$ i.e $\bar{\mathfrak{h}}_n$ acts on \mathfrak{I}_n via matrix multiplication on vectors. More precisely, \mathfrak{I}_n is an abelian ideal of \mathfrak{sch}_n acting on \mathfrak{sch}_n via the following brackets of vector fields:

$$[X_{ij}, x_i \frac{\partial}{\partial x^{n+1}}] = x_j \frac{\partial}{\partial x^{n+1}}, \quad [X_{ij}, x_i \frac{\partial}{\partial x^{n+2}}] = x_j \frac{\partial}{\partial x^{n+2}}, \quad [a_n, x_i \frac{\partial}{\partial x^{n+1}}] = x_i \frac{\partial}{\partial x^{n+1}},$$

$$[a_n, x_i \frac{\partial}{\partial x^{n+2}}] = -x_i \frac{\partial}{\partial x^{n+2}}, \quad [b_n, x_i \frac{\partial}{\partial x^{n+1}}] = -x_i \frac{\partial}{\partial x^{n+2}}, \quad [b_n, x_i \frac{\partial}{\partial x^{n+2}}] = 0,$$

$$[c_n, x_i \frac{\partial}{\partial x^{n+1}}] = 0, \quad [c_n, x_i \frac{\partial}{\partial x^{n+2}}] = x_i \frac{\partial}{\partial x^{n+1}}, \quad [x_i \frac{\partial}{\partial x^{n+1}}, x_j \frac{\partial}{\partial x^{n+2}}] = 0.$$

In this framework, $X_{ij}, a_n, b_n, c_n, x_i \frac{\partial}{\partial x^{n+1}}$ and $x_i \frac{\partial}{\partial x^{n+2}}$ represent respectively the generators of the rotations, dilation, time translation (Hamiltonian), conformal transformation, Galilean boosts and space translations (momentum operators).

Also the Lie algebra $\bar{\mathfrak{h}}_n$ acts on \mathfrak{I}_n and \mathfrak{sch}_n via the bracket of vector fields. This action is extended to $\mathfrak{I}_n^{\wedge k}$ by

$$[\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k, X] = \sum_{i=1}^k \alpha_1 \wedge \alpha_2 \wedge \dots \wedge [\alpha_i, X] \wedge \dots \wedge \alpha_k$$

for $\alpha_i \in \mathfrak{I}_n$, $X \in \bar{\mathfrak{h}}_n$, and the action of $\bar{\mathfrak{h}}_n$ on $\mathfrak{sch}_n \otimes \mathfrak{I}_n^{\wedge k}$ is given by

$$\begin{aligned} [g \otimes \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k, X] &= [g, X] \otimes \alpha_1 \wedge \dots \wedge \alpha_k \\ &+ \sum_{i=1}^k g \otimes \alpha_1 \wedge \alpha_2 \wedge \dots \wedge [\alpha_i, X] \wedge \dots \wedge \alpha_k \end{aligned}$$

for $g \in \mathfrak{sch}_n$.

For the remaining of the paper, we write $\mathfrak{so}(n)$, $\mathfrak{gl}(2)$ and $\mathfrak{sl}(2)$ for $\mathfrak{so}(n; \mathbb{R})$, $\mathfrak{gl}(2; \mathbb{R})$ and $\mathfrak{sl}(2; \mathbb{R})$ respectively.

Lemma 3.2. *There is a graded vector space isomorphism:*

$$[\wedge^*(\mathfrak{I}_n)]^{\bar{\mathfrak{h}}_n} \cong \mathbb{R} \oplus \langle \beta_n \rangle \oplus \langle \zeta_n \rangle \oplus \langle \alpha_n \rangle$$

where

$$\alpha_n = y_1 \wedge \dots \wedge y_n \wedge y_{n+1} \wedge \dots \wedge y_{2n}, \quad \beta_n = \sum_{i=1}^n y_i \wedge y_{n+i}$$

and

$$\zeta_n = \sum_{i=1}^n y_1 \wedge \dots \wedge \widehat{y_i} \dots \wedge y_n \wedge y_{n+1} \wedge \dots \wedge \widehat{y_{n+i}} \dots \wedge y_{2n}$$

with $y_i := x_i \frac{\partial}{\partial x^{n+1}}$ and $y_{n+i} := x_i \frac{\partial}{\partial x^{n+2}}$ for $1 \leq i \leq n$.

Proof. Clearly $[\mathbb{R}]^{\bar{\mathfrak{h}}_n} = \mathbb{R}$. Now let $\omega \in \mathfrak{I}_n$. Then $\omega = \sum_{1 \leq i \leq n} c_i x_i \frac{\partial}{\partial x^{n+1}} + \sum_{1 \leq i \leq n} c'_i x_i \frac{\partial}{\partial x^{n+2}}$ for real constants c_i, c'_i . Assume without loss of generality that $c_{i_0} \neq 0$ for some $i_0 \neq n$. Then $[\omega, X_{i_0 n}] = -c_{i_0} x_n \frac{\partial}{\partial x^{n+1}} + c_n x_{i_0} \frac{\partial}{\partial x^{n+1}} \neq 0$. So $\omega \notin [\mathfrak{I}_n]^{\bar{\mathfrak{h}}_n}$ and thus $[\mathfrak{I}_n]^{\bar{\mathfrak{h}}_n} = 0$. Now write $\mathfrak{I}_n = \mathfrak{I}_n^1 \oplus \mathfrak{I}_n^2$ with $\mathfrak{I}_n^1 = \langle y_1, \dots, y_n \rangle$ and $\mathfrak{I}_n^2 = \langle y_{n+1}, \dots, y_{2n} \rangle$. Then since $\mathfrak{so}(n)$ is a Lie subalgebra of $\bar{\mathfrak{h}}_n$, it follows by [1, lemma 4.1] that

$$[\wedge^k(\mathfrak{I}_n^i)]^{\bar{\mathfrak{h}}_n} \subseteq [\wedge^k(\mathfrak{I}_n^i)]^{\mathfrak{so}(n)} = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ \langle x_1 \frac{\partial}{\partial x^{n+i}} \wedge \dots \wedge x_n \frac{\partial}{\partial x^{n+i}} \rangle & \text{if } k = n \\ 0 & \text{else} \end{cases}$$

for $i = 1, 2$. However, $x_1 \frac{\partial}{\partial x^{n+i}} \wedge \dots \wedge x_n \frac{\partial}{\partial x^{n+i}} \notin [\wedge^k(\mathfrak{I}_n^i)]^{\bar{\mathfrak{h}}}$ because

$$[x_1 \frac{\partial}{\partial x^{n+i}} \wedge \dots \wedge x_n \frac{\partial}{\partial x^{n+i}}, a_n] = (-1)^i n x_1 \frac{\partial}{\partial x^{n+i}} \wedge \dots \wedge x_n \frac{\partial}{\partial x^{n+i}} \neq 0.$$

Now let $\omega = \sum_{A_*, B_*} c_{**} A_* \wedge B_* \in [(\mathfrak{I}_n^1)^{\wedge r} \wedge (\mathfrak{I}_n^2)^{\wedge s}]^{\bar{\mathfrak{h}}_n}$ with $A_* \in (\mathfrak{I}_n^1)^{\wedge r}$ and $B_* \in (\mathfrak{I}_n^2)^{\wedge s}$.

If $r \neq s$, we have $[\omega, a_n] = [\sum_{A_*, B_*} c_{**} A_* \wedge B_*, a_n] = (s - r)\omega \neq 0$. This is a contradiction.

If $r = s = 1$, one easily shows that

$$[(\mathfrak{I}_n^1) \wedge (\mathfrak{I}_n^2)]^{\bar{\mathfrak{h}}_n} = \langle \beta_n \rangle.$$

If $r = s = n - 1$, one also shows that

$$[(\mathfrak{J}_{n-1}^1)^{n-1} \wedge (\mathfrak{J}_{n-1}^2)^{n-1}]^{\bar{h}_n} = \langle \zeta_n \rangle.$$

If $r = s = n$, a straightforward calculation shows that

$$[(\mathfrak{J}_n^1)^{\wedge n} \wedge (\mathfrak{J}_n^2)^{\wedge n}]^{\bar{h}_n} = \langle \alpha_n \rangle.$$

For $1 < r = s < n - 1$, we show that $[(\mathfrak{J}_n^1)^{\wedge s} \wedge (\mathfrak{J}_n^2)^{\wedge s}]^{\bar{h}_n} = 0$ by showing by induction on n that $[(\mathfrak{J}_n^1)^{\wedge s} \wedge (\mathfrak{J}_n^2)^{\wedge s}]^{\mathfrak{so}(n)} = 0$. Indeed, it is easy to check the result for $n = 4$ and $r = s = 2$. By the inductive hypothesis, suppose $[(\mathfrak{J}_{n-1}^1)^{\wedge s} \wedge (\mathfrak{J}_{n-1}^2)^{\wedge s}]^{\mathfrak{so}(n-1)} = 0$ for $s \neq 0, 1, n-2, n-1$ and let $z \in [(\mathfrak{J}_n^1)^{\wedge s} \wedge (\mathfrak{J}_n^2)^{\wedge s}]^{\mathfrak{so}(n)}$ with $s \neq 0, 1, n-1, n$ fixed. Then

$$z = c_1 A_1 \wedge B_1 + c_2 A_2 \wedge B_2 \wedge y_n + c_3 A_3 \wedge B_3 \wedge y_{2n} + c_4 A_4 \wedge B_4 \wedge y_n \wedge y_{2n}$$

where $A_1, A_3 \in (\mathfrak{J}_{n-1}^1)^{\wedge s}$, $B_1, B_2 \in (\mathfrak{J}_{n-1}^2)^{\wedge s}$, $A_2, A_4 \in (\mathfrak{J}_{n-1}^1)^{\wedge s-1}$, $B_3, B_4 \in (\mathfrak{J}_{n-1}^2)^{\wedge s-1}$, and $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Let $X \in \mathfrak{so}(n-1) \subseteq \mathfrak{so}(n)$ as a Lie subalgebra, we have

$$0 = [z, X] = c_1 [A_1 \wedge B_1, X] + c_2 [A_2 \wedge B_2, X] \wedge y_n + c_3 [A_3 \wedge B_3, X] \wedge y_{2n} + c_4 [A_4 \wedge B_4, X] \wedge y_n \wedge y_{2n}.$$

If the coefficients are non-zero, then the terms $[A_1 \wedge B_1, X]$, $[A_2 \wedge B_2, X]$, $[A_3 \wedge B_3, X]$ and $[A_4 \wedge B_4, X]$ are all zero By linear independence. This implies that

$$A_2 \wedge B_2 \in [(\mathfrak{J}_{n-1}^1)^{\wedge s-1} \wedge (\mathfrak{J}_{n-1}^2)^{\wedge s}]^{\mathfrak{so}(n-1)} = 0, \quad A_3 \wedge B_3 \in [(\mathfrak{J}_{n-1}^1)^{\wedge s} \wedge (\mathfrak{J}_{n-1}^2)^{\wedge s-1}]^{\mathfrak{so}(n-1)} = 0$$

by the case $r \neq s$ above, and

$$A_1 \wedge B_1 \in [(\mathfrak{J}_{n-1}^1)^{\wedge s} \wedge (\mathfrak{J}_{n-1}^2)^{\wedge s}]^{\mathfrak{so}(n-1)} = 0, \quad A_4 \wedge B_4 \in [(\mathfrak{J}_{n-1}^1)^{\wedge s-1} \wedge (\mathfrak{J}_{n-1}^2)^{\wedge s-1}]^{\mathfrak{so}(n-1)} = 0$$

by inductive hypothesis. Hence $z = 0$. \square

Lemma 3.3. *For all integer k ,*

$$[\mathfrak{sl}(2) \otimes \mathfrak{J}_n^{\wedge k}]^{\bar{h}_n} = 0$$

Proof. Let k_1 and k_2 with $0 \leq k_1, k_2 \leq n$ and let $\omega \in [\mathfrak{sl}(2) \otimes (\mathfrak{J}_n^1)^{\wedge k_1} \wedge (\mathfrak{J}_n^2)^{\wedge k_2}]^{\bar{h}}$. Then

$$\omega = \sum_{A_*, B_*} c_a^{**} a_n \otimes A_* \wedge B_* + \sum_{A_*, B_*} c_b^{**} b_n \otimes A_* \wedge B_* + \sum_{A_*, B_*} c_c^{**} c_n \otimes A_* \wedge B_*$$

where $A_* \in (\mathfrak{J}_n^1)^{\wedge k_1}$, $B_* \in (\mathfrak{J}_n^2)^{\wedge k_2}$, and $c_a^{**}, c_b^{**}, c_c^{**}$ are real coefficients.

$$\begin{aligned} 0 = [\omega, a_n] &= (k_2 - k_1) \sum_{A_*, B_*} c_a^{**} a_n \otimes A_* \wedge B_* \\ &\quad + (k_2 - k_1 + 2) \sum_{A_*, B_*} c_b^{**} b_n \otimes A_* \wedge B_* + (k_2 - k_1 - 2) \sum_{A_*, B_*} c_c^{**} c_n \otimes A_* \wedge B_*. \end{aligned}$$

If $k_2 \neq k_1, k_1 - 2, k_1 + 2$, this implies by linear independence that all the coefficients $c_a^{**}, c_b^{**}, c_c^{**}$ are zero and thus $\omega = 0$. However if $k_2 = k_1$, only the coefficients c_b^{**}, c_c^{**} are zero by linear independence. So $\omega = \sum_{A_*, B_*} c_a^{**} a_n \otimes A_* \wedge B_*$. Now

$$0 = [\omega, c_n] = 2 \sum_{A_*, B_*} c_a^{**} c_n \otimes A_* \wedge B_* + \sum_{A_*, B_*} c_a^{**} a_n \otimes A_* \wedge [B_*, c_n].$$

This implies that the coefficients c_a^{**} are zero by linear independence. Hence $\omega = 0$.

If $k_2 = k_1 - 2$, only the coefficients c_a^{**}, c_c^{**} are zero by linear independence.

So $\omega = \sum_{A_*, B_*} c_b^{**} b_n \otimes A_* \wedge B_*$. Now

$$0 = [\omega, c_n] = \sum_{A_*, B_*} c_b^{**} a_n \otimes A_* \wedge B_* + \sum_{A_*, B_*} c_b^{**} b_n \otimes A_* \wedge [B_*, c_n].$$

This implies that the coefficients c_b^{**} are zero by linear independence. Hence $\omega = 0$.

Similarly, if $k_2 = k_1 + 2$, only the coefficients c_a^{**}, c_b^{**} are zero by linear independence. Now the condition $[\omega, c_n] = 0$ annihilates the coefficients c_c^{**} by linear independence. Hence $\omega = 0$. \square

Lemma 3.4. *The following are vector space isomorphisms*

$$[\mathfrak{so}(n) \otimes \mathfrak{J}_n^{\wedge k}]^{\bar{\mathfrak{h}}_n} = \begin{cases} 0, & \text{if } k \neq 2, n-2 \\ \langle \rho_n \rangle, & \text{if } k = 2 \\ \langle \gamma_n \rangle, & \text{if } k = n-2 \end{cases}$$

where

$$\rho_n = \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_i \wedge y_{n+j} - \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_j \wedge y_{n+i}$$

and

$$\begin{aligned} \gamma_n &= \sum_{1 \leq i < j \leq n} X_{ij} \otimes (y_1 \wedge \dots \widehat{y_i} \dots \wedge y_n \wedge y_{n+1} \wedge \dots \widehat{y_{n+j}} \dots \wedge y_{2n}) \\ &\quad - \sum_{1 \leq i < j \leq n} X_{ij} \otimes (y_1 \wedge \dots \widehat{y_j} \dots \wedge y_n \wedge y_{n+1} \wedge \dots \widehat{y_{n+i}} \dots \wedge y_{2n}) \end{aligned}$$

Proof. Clearly, $[\mathfrak{so}(n)]^{\bar{\mathfrak{h}}_n} = 0$ since $[X_{ij}, X_{ik}] = X_{jk} \neq 0$. Also since $\mathfrak{so}(n)$ is a subalgebra of $\bar{\mathfrak{h}}_n$, it follows by [1, lemma 4.2] that $[\mathfrak{so}(n) \otimes \wedge^k(\mathfrak{J}_n^1)]^{\bar{\mathfrak{h}}_n}$ is a submodule of

$$[\mathfrak{so}(n) \otimes \wedge^k(\mathfrak{J}_n^1)]^{\mathfrak{so}(n)} = \begin{cases} 0, & \text{if } k \neq 2, n-2 \\ \langle \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_i \wedge y_j \rangle, & \text{if } k = 2 \\ \langle \sum_{1 \leq i < j \leq n} \text{sgn}(\sigma_{ij}) X_{ij} \otimes y_1 \wedge \dots \widehat{y_i} \dots \widehat{y_j} \dots \wedge y_n \rangle, & \text{if } k = n-2. \end{cases}$$

However, $[\sum_{1 \leq i < j \leq n} X_{ij} \otimes y_i \wedge y_j, b_n] = \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_{n+i} \wedge y_j + \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_i \wedge y_{n+j} \neq 0$.

Also, one shows that $\sum_{1 \leq i < j \leq n} X_{ij} \otimes y_1 \wedge \dots \widehat{y_i} \dots \widehat{y_j} \dots \wedge y_n \notin [\mathfrak{so}(n) \otimes \wedge^{n-1}(\mathfrak{J}_n^1)]^{\bar{\mathfrak{h}}_n}$. Hence

$[\mathfrak{so}(n) \otimes \wedge^k(\mathfrak{J}_n^1)]^{\bar{\mathfrak{h}}_n} = 0$ for all k . similarly, one shows that $[\mathfrak{so}(n) \otimes \wedge^k(\mathfrak{J}_n^2)]^{\bar{\mathfrak{h}}_n} = 0$ using c_n . Now let r and s with $1 \leq r, s \leq n$ and let $\omega \in [\mathfrak{so}(n) \otimes (\mathfrak{J}_n^1)^{\wedge r} \wedge (\mathfrak{J}_n^2)^{\wedge s}]^{\bar{\mathfrak{h}}_n}$. Then

$$\omega = \sum_{1 \leq i < j \leq n} c_{ij}^{**} X_{ij} \otimes A_* \wedge B_*$$

with $A_* \in (\mathfrak{J}_n^1)^{\wedge r}$, $B_* \in (\mathfrak{J}_n^2)^{\wedge s}$. If $r \neq s$, we have

$$0 = [\omega, a_n] = (s-r) \sum_{\substack{1 \leq i < j \leq n \\ A_*, B_*}} c_{ij}^{**} X_{ij} \otimes A_* \wedge B_*$$

as $[A_*, a_n] = -rA_*$ and $[B_*, a_n] = sB_*$. So all the c_{ij}^{**} s are zero by linear independence. So $\omega = 0$. For the case $r = s$, first notice that since $\mathfrak{so}(n)$ is a Lie subalgebra of $\bar{\mathfrak{h}}_n$, it follows that $[\mathfrak{so}(n) \otimes \wedge^k(\mathfrak{J}_n)]^{\bar{\mathfrak{h}}} \subseteq [\mathfrak{so}(n) \otimes \wedge^k(\mathfrak{J}_n)]^{\mathfrak{so}(n)}$. Now following again the proof of [1, lemma 4.2], we have that

$$\dim[\mathfrak{so}(n) \otimes \mathfrak{J}_n^{\wedge k}]^{\mathfrak{so}(n)} = \dim \text{Hom}_{\mathfrak{so}(n)}(\mathfrak{J}_n^{\wedge 2}, \mathfrak{J}_n^{\wedge k}) = \begin{cases} 1, & \text{if } k = 2, 2n-2 \\ 0, & \text{else} \end{cases}.$$

So two cases remain here for possible $\bar{\mathfrak{h}}_n$ -invariants: Firstly, if $r = s = 1$ we have

$$\omega = \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k, l \leq n}} c_{ij}^{kl} X_{ij} \otimes x_k \frac{\partial}{\partial x^{n+1}} \wedge x_l \frac{\partial}{\partial x^{n+2}}. \text{ However since } [\omega, b_n] = [\omega, c_n] = 0, \text{ it follows by}$$

linear independence that all coefficients c_{ij}^{kl} , $k \neq l$ are zero except for $(i, j) = (k, l)$ in which case successive choices of $X_{ij} \in \mathfrak{so}(n)$ with $[X_{ij}, \omega] = 0$ imply by linear independence that $c_{ij}^{ij} = -c_{ji}^{ji} = c$ for some constant c . Thus

$$\omega = c \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_i \wedge y_{n+j} - \sum_{1 \leq i < j \leq n} X_{ij} \otimes y_j \wedge y_{n+i}.$$

Secondly, if $r = s = n-1$, the proof that $\omega = c\gamma_n$ for some constant c is similar to the first case. \square

Lemma 3.5. *For all integer k ,*

$$[\mathfrak{J}_n \otimes \mathfrak{J}_n^{\wedge k}]^{\bar{\mathfrak{h}}_n} = 0.$$

Proof. Write $\mathfrak{J}_n = \mathfrak{J}_n^1 \oplus \mathfrak{J}_n^2$ as in the proof of lemma 3.2. Note that since $\mathfrak{so}(n)$ is a Lie subalgebra of $\bar{\mathfrak{h}}_n$, it follows by [1, lemma 4.3] that $[\mathfrak{J}_n^i \otimes \wedge^k(\mathfrak{J}_n^i)]^{\bar{\mathfrak{h}}_n} \subseteq [\mathfrak{J}_n^i \otimes \wedge^k(\mathfrak{J}_n^i)]^{\mathfrak{so}(n)} = 0$ for all $k \neq 1, n-1$, for $i = 1, 2$. Now by [1, lemma 4.4],

$$[\mathfrak{J}_n^1 \otimes \mathfrak{J}_n^1]^{\bar{\mathfrak{h}}_n} \subseteq [\mathfrak{J}_n^1 \otimes \mathfrak{J}_n^1]^{\mathfrak{so}(n)} = \left\langle \sum_{i=1}^n y_i \otimes y_i \right\rangle$$

and by [1, lemma 4.5],

$$[\mathfrak{J}_n^1 \otimes \wedge^{n-1}(\mathfrak{J}_n^1)]^{\bar{\mathfrak{h}}_n} \subseteq [\mathfrak{J}_n^1 \otimes \wedge^{n-1}(\mathfrak{J}_n^1)]^{\mathfrak{so}(n)} = \left\langle \sum_{m=1}^n (-1)^{m-1} y_m \otimes y_1 \wedge y_2 \dots \widehat{y_m} \dots \wedge y_n \right\rangle.$$

However

$$\left[\sum_{i=1}^n y_i \otimes y_i, b_n \right] = - \sum_{i=1}^n (y_i \otimes y_{n+i} + y_{n+i} \otimes y_i) \neq 0$$

and

$$\left[\sum_{m=1}^n (-1)^{m-1} y_m \otimes y_1 \wedge \dots \widehat{y_m} \dots \wedge y_n, b_n \right] = - \sum_{m=1}^n (-1)^m y_{n+m} \otimes y_1 \wedge \dots \widehat{y_m} \dots \wedge y_n + \sum_{i=1}^n \sum_{m=1}^n (-1)^{m-1} y_m \otimes y_1 \wedge \dots \wedge y_{n+i} \wedge \dots \widehat{y_m} \dots \wedge y_n \neq 0. \text{ So}$$

$$[\mathfrak{J}_n^1 \otimes \wedge^{n-1}(\mathfrak{J}_n^1)]^{\bar{\mathfrak{h}}_n} = 0 \text{ and } [\mathfrak{J}_n^1 \otimes \mathfrak{J}_n^1]^{\bar{\mathfrak{h}}_n} = 0.$$

Similarly,

$$[\mathfrak{J}_n^2 \otimes \wedge^{n-1}(\mathfrak{J}_n^2)]^{\bar{\mathfrak{h}}_n} = 0 \text{ and } [\mathfrak{J}_n^2 \otimes \mathfrak{J}_n^2]^{\bar{\mathfrak{h}}_n} = 0.$$

Now in general, let k_1 and k_2 with $1 \leq k_1, k_2 \leq n$ and let $\omega \in [\mathfrak{J}_n \otimes (\mathfrak{J}_n^1)^{\wedge k_1} \wedge (\mathfrak{J}_n^1)^{\wedge k_2}]^{\bar{\mathfrak{h}}_n}$. Then $\omega = \sum_{\substack{1 \leq i \leq n \\ A_*, B_*}} c_i^{**} y_i \otimes A_* \wedge B_* + \sum_{\substack{1 \leq i \leq n \\ A_*, B_*}} c_{n+i}^{**} y_{n+i} \otimes A_* \wedge B_*$ with $A_* \in (\mathfrak{J}_n^1)^{\wedge k_1}$, $B_* \in (\mathfrak{J}_n^2)^{\wedge k_2}$ and c_i^{**} , c_{n+i}^{**} real coefficients. We have

$$0 = [\omega, a_n] = (k_2 - k_1 - 1) \sum_{A_*, B_*} c_i^{**} y_i \otimes A_* \wedge B_* + (k_2 - k_1 + 1) \sum_{A_*, B_*} c_{n+i}^{**} y_{n+i} \otimes A_* \wedge B_*.$$

Two cases occur: Firstly, if $k_2 = k_1 + 1$, all the coefficients c_{n+i}^{**} are zero. Now since

$$0 = [\omega, b_n] = \sum_{\substack{1 \leq i \leq n \\ A_*, B_*}} c_i^{**} y_{n+i} \otimes A_* \wedge B_* + \sum_{\substack{1 \leq i \leq n \\ A_*, B_*}} c_i^{**} y_i \otimes [A_*, b_n] \wedge B_*,$$

it follows that all the coefficients c_i^{**} are zero by linear independence. So $\omega = 0$. Secondly, if $k_2 = k_1 - 1$, all the coefficients c_i^{**} are zero. Similarly, the condition $0 = [\omega, c_n]$ annihilates all the coefficients c_{n+i}^{**} by linear independence. So $\omega = 0$. \square

As a consequence of these lemmas we have the following:

Theorem 3.6. *There are graded vector space isomorphisms*

$$H_*^{Lie}(\mathfrak{sch}_n; \mathbb{R}) \cong H_*^{Lie}(\mathfrak{sl}(2); \mathbb{R}) \otimes H_*^{Lie}(\mathfrak{so}(n); \mathbb{R}) \otimes (\mathbb{R} \oplus \langle \zeta_n \rangle \oplus \langle \alpha_n \rangle),$$

and

$$HL_*(\mathfrak{sch}; \mathbb{R}) \cong (\mathbb{R} \oplus \langle \tilde{\zeta}_n \rangle \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n),$$

where $\tilde{\alpha}_n = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) y_{\sigma(1)} \otimes \dots \otimes y_{\sigma(n)} \otimes y_{\sigma(n+1)} \otimes \dots \otimes y_{\sigma(2n)}$ is the antisymmetrization of α_n ,

$$\tilde{\zeta}_n = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n-2}} \text{sgn}(\sigma) y_{\sigma(1)} \otimes \dots \otimes \widehat{y_{\sigma(i)}} \dots \otimes y_{\sigma(n)} \otimes y_{\sigma(n+1)} \otimes \dots \otimes \widehat{y_{\sigma(n+i)}} \dots \otimes y_{\sigma(2n)}$$

is the antisymmetrization of ζ_n and $\tilde{\gamma}_n$ is the $\bar{\mathfrak{h}}_n$ -invariant cycle in $\mathfrak{sch}_n^{\otimes(n-1)}$ representing γ_n .

Proof. The first isomorphism follows by lemma 2.1, lemma 3.2 and the Künneth formula for Lie algebra homology. Note that the $\bar{\mathfrak{h}}_n$ -invariants β_n is zero in $H_*^{Lie}(\mathfrak{sch}_n; \mathbb{R})$ since

$$d(\bar{\rho}_n) = -2(n-1)\beta_n$$

where d is the Chevalley-Eilenberg boundary map and

$$\bar{\rho}_n = \sum_{1 \leq i < j \leq n} X_{ij} \wedge y_i \wedge y_{n+j} - \sum_{1 \leq i < j \leq n} X_{ij} \wedge y_j \wedge y_{n+i}.$$

For the second isomorphism, notice that γ_n and ρ_n are the only $\bar{\mathfrak{h}}_n$ -invariants in $\mathfrak{sch}_n \otimes \mathfrak{J}_n^{\wedge*}$. But ρ_n is not a cycle in $H_*^{Lie}(\mathfrak{J}_n; \mathfrak{sch}_n)^{\bar{\mathfrak{h}}_n}$ as it maps to

$$d(\rho_n) = -2(n-1) \sum_{i=1}^n y_i \otimes y_{n+i} \neq 0$$

by the Loday boundary map d . So the kernel K_* of the composition

$$\pi_* \circ j_* : H_*^{Lie}(\mathfrak{J}_n; \mathfrak{sch}_n)^{\bar{h}_n} \rightarrow H_*^{Lie}(\mathfrak{sch}_n; \mathfrak{sch}_n) \rightarrow H_{*+1}^{Lie}(\mathfrak{sch}_n; \mathbb{R})$$

is $K_* = \langle \gamma_n \rangle$. Note that $\pi_* \circ j_* \circ d(\rho_n) = -2(n-1)\beta_n \neq 0$ and $\pi_* \circ j_*(\gamma_n) = 0$ in $H_{*+1}^{Lie}(\mathfrak{sch}_n; \mathbb{R})$. Now since \mathfrak{h}_n is a semisimple Lie algebra, we have by Lodder's structure theorem [10, Lemma 3.6] that

$$HL_*(\mathfrak{sch}_n; \mathbb{R}) \cong [\wedge^*(\mathfrak{J}_n)]^{\bar{h}_n} \otimes T(K_*) \cong (\mathbb{R} \oplus \langle \tilde{\zeta}_n \rangle \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n).$$

(The construction of $\tilde{\gamma}_n$ is similar to [1, lemma 4.6]). □

4 Leibniz homology of the full Galilei algebra

We now turn back our attention to the full Galilei algebra $\widetilde{\mathfrak{gal}}_n$. From the matrix (M.1), we can construct the Lie algebra isomorphisms

$$\mathfrak{gl}(2; \mathbb{R}) \cong \text{Span}\{a_n, b_n, c_n, d_n\} \quad \text{and} \quad \widetilde{\mathfrak{gal}}_n \cong \mathfrak{sch}_n \oplus \langle d_n \rangle$$

where

$$d_n := x_{n+1} \frac{\partial}{\partial x^{n+1}} + x_{n+2} \frac{\partial}{\partial x^{n+2}},$$

and in addition to the brackets in the Lie algebra \mathfrak{sch}_n , we have the following brackets

$$[X_{ij}, d_n] = 0, \quad [a_n, d_n] = 0, \quad [b_n, d_n] = 0, \quad [c_n, d_n] = 0.$$

$$[d_n, x_i \frac{\partial}{\partial x^{n+1}}] = -x_i \frac{\partial}{\partial x^{n+1}} \quad [d_n, x_i \frac{\partial}{\partial x^{n+2}}] = -x_i \frac{\partial}{\partial x^{n+2}}.$$

Corollary 4.1. *There is a graded vector space isomorphism*

$$HL_*(\widetilde{\mathfrak{gal}}_n; \mathbb{R}) \cong ((\mathbb{R} \oplus \langle \tilde{\zeta}_n \rangle \oplus \langle \tilde{\alpha}_n \rangle) \otimes T^*(\tilde{\gamma}_n)) * T^*(\mathbb{R})$$

where $*$ is the non-commutative tensor product of \mathbb{N} -graded modules.

Proof. Since $\widetilde{\mathfrak{gal}}_n \cong \mathfrak{sch}_n \oplus \langle d_n \rangle$ and $H_*^{Lie}(\langle d_n \rangle; \mathbb{R}) \cong T^*(\mathbb{R})$, the result follows by [8, theorem 3] and theorem 3.6. □

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